1 Equations

Note on time steps: webDICE, following DICE, is implemented in 10-year time steps. The equations are presented for arbitrary intervals for ease of readabilty. Most of the data is in annual time steps and is converted to 10-year units within the model.

Objective function

1. $W = \sum_{t=1}^{Tmax} L(t)U(t)R(t)$
2. $R(t) = 1/[(1+\rho)^t]$
3. $U(t) = c(t)^{1-\alpha}/(1-\alpha) + 1$
4. $c(t) = C(t)/L(t)$

Population function

5. $L(t) = [L(t-1) \times L(Tmax)]^{0.5}$

Production function

6. $Q(t) = [1-\Omega(t)][1-\Lambda(t)]Y(t)$
7. $Y(t) = A(t)K(t)^{\gamma}L(t)^{1-\gamma}$
8. $C(t) = Q(t) - I(t)$
9. $I(t) = s \times Q(t)$
10. $K(t) = I(t) + (1-\delta_K)K(t-1)$

Total Factor Productivity

11. $A(t) = A(t-1)/[1-A_g(t-1)]$
12. $A_g(t) = A_g(0) \times exp\{-\Delta_a \cdot t \times exp[-\Delta_b \cdot t]\}$

Climate damage functions

Default Damages

13. $\Omega(t) = 1 - 1/[1 + \pi_2 T_{AT}(t)^{\epsilon}]$

Environmental Goods

14. $\Omega(t) = 1/[1 + C_{EG}(t) \times \pi_{EG} T_{AT}(t)^{\epsilon}]$, with

$$\pi_{EG} = \frac{\pi_2}{C(x)} \text{ where } T_{AT}(x) = 2.5°\text{C}$$

15. $C_{EG} = (1 - s)[1 - \Lambda(t)]Y(t)$

Tipping Point

16. $\Omega(t) = 1 - 1/[1 + (T_{AT}/20.46)^2 + (T_{AT}/6.081)^{6.754}]$

Damages to productivity growth

17. $A(t) = (1 - f \cdot \Omega_{def}(t)) \times [A(t-1)/(1 - A_g(t-1)]$

18. $\Omega(t) = 1 - [1 - \Omega_{def}(t)]/[1 - f \cdot \Omega_{def}(t)]$

19. $\Omega_{def}(t) = 1 - 1/[1 + \pi_2 T_{AT}(t)^{\epsilon}]$

Abatement cost function

20. $\Lambda(t) = \varphi(t)^{1-\theta_2} BC(t)\mu(t)^{\theta_2}\sigma(t)/\theta_2$

21. $$\varphi(t) = \begin{cases} \varphi(5) + [\varphi(0) - \varphi(5)] \times exp(-0.25 \cdot t) & if \quad t < 5 \\ \varphi(10) + [\varphi(5) - \varphi(10)] \times exp(-0.25 \cdot t) & if \quad 5 \le t < 10 \\ \varphi(15) + [\varphi(10) - \varphi(15)] \times exp(-0.25 \cdot t) & if \quad 10 \le t < 15 \\ \varphi(15) + [\varphi_{MAX} - \varphi(15)] \times exp(-0.25 \cdot t) & if \quad 15 \le t \end{cases}$$

22. $BC(t) = BC(0) \times (1 - BC_g)^t$

23. $$\mu(t) = \begin{cases} 0 & if \quad E_{ind}(t) \le E_{ind}(0) \times ecap(t) \\ 1 - \frac{E_{ind}(0) \times ecap}{\sigma(t)A(t)K(t)^{\gamma}L(t)^{1-\gamma}} & if \quad E_{ind}(t) > E_{ind}(0) \times ecap(t) \end{cases}$$

Emissions

24. $CCum \geq \sum_{t=0}^{Tmax} E(t)$

25. $E(t) = E_{Ind}(t) + E_{Land}(t)$

26. $E_{Ind}(t) = \sigma(t)[1 - \mu(t)]A(t)K(t)^{\gamma}L(t)^{1-\gamma}$

27. $E_{Land}(t) = E_{Land}(0) \times (0.8)^t$

Carbon intensity

28. $\sigma(t) = \sigma(t-1) \times (1 - \sigma_g(t-1))$

29. $\sigma_g(t) = \sigma_g(t-1) \times (1 - \sigma_{d1})$

Carbon tax

30. $\tau(t) = BC(t)\mu(t)^{\theta_2 - 1}$, for $\varphi = 1$.

Carbon cycle and Climate Model

31. $$\begin{bmatrix} M_{AT}(t) \\ M_{UP}(t) \\ M_{LO}(t) \end{bmatrix} = \begin{bmatrix} \varphi_{11} & \varphi_{12} & 0 \\ 1-\varphi_{11} & 1-\varphi_{12}-\varphi_{32} & \varphi_{23} \\ 0 & \varphi_{32} & 1-\varphi_{23} \end{bmatrix} \begin{bmatrix} M_{AT}(t-1) \\ M_{UP}(t-1) \\ M_{LO}(t-1) \end{bmatrix} + \begin{bmatrix} E(t-1) \\ 0 \\ 0 \end{bmatrix}$$

Climate Model

32. $T_{AT}(t) = T_{AT}(t-1) + \xi_1\{F(t) - \lambda T_{AT}(t-1) - \xi_2[T_{AT}(t-1) - T_{LO}(t-1)]\}$, where $\lambda = \frac{F_{2\times CO_2}}{T_{2\times CO_2}}$

33. $T_{LO}(t) = T_{LO}(t-1) + \xi_3\{T_{AT}(t-1) - T_{LO}(t-1)\}$

34. $F(t) = \eta\{log_2[M_{AT}(t)/M_{PI}]\} + F_{EX}(t)$

35. $$F_{EX}(t) = \begin{cases} F_{EX}(0) + 0.1\,(F_{EX}(10) - F_{EX}(0)) \cdot t & if \quad t \leq 10 \\ F_{EX}(10) & if \quad t > 10 \end{cases}$$

Alternative Carbon Cycle: Simplified BEAM

36. $$\frac{d}{dt}\begin{bmatrix} M_{AT}(t) \\ M_{UP}(t) \\ M_{LO}(t) \end{bmatrix} = \begin{bmatrix} -k_a & k_a \cdot A \cdot B & 0 \\ k_a & -(k_a \cdot A \cdot B) - k_d & \frac{k_d}{\delta} \\ 0 & k_d & -\frac{k_d}{\delta} \end{bmatrix} \begin{bmatrix} M_{AT} \\ M_{UP} \\ M_{LO} \end{bmatrix} + E(t)$$

37. $A = k_H \cdot \frac{AM}{OM/(\delta+1)}$

38. $B = \frac{1}{1 + \frac{k_1}{[H^+]} + \frac{k_1 k_2}{[H^+]^2}}$

39. $\frac{M_{UP}}{Alk} = \frac{1 + \frac{k_1}{[H^+]} + \frac{k_1 k_2}{[H^+]^2}}{\frac{k_1}{[H^+]} + \frac{2k_1 k_2}{[H^+]^2}}$

Alternative Climate Model: Linear temperature

40. $T_{AT}(t) = T_{AT}(0) + \sum_{t=0}^{t} E(t)\{T_{2\times CO_2}/[\psi \times 2M_{PI} - \phi_{11}M_{AT}(0) - \phi_{21}M_{UP}(0)]\}$

2 Variables

$U(t)$: social utility

$R(t)$: social time preference discount factor

$\rho(t)$: social time preference rate

$C(t)$: total consumption, trillions of 2005 US dollars

$c(t)$: per capita consumption, thousands of 2005 US dollars

$I(t)$: investment

$K(t)$: capital, trillions of 2005 US dollars

$A(t)$: total factor productivity

$E(t)$: total carbon emissions, GtC per period

$E_{Ind}(t)$: industrial carbon emissions, GtC per period

$T_{AT}(t)$: global mean surface temperature, °C increase from 1900

$T_{LO}(t)$: global mean deep ocean temperature, °C increase from 1900

$F(t)$: radiative forcings due to CO_2, W/m^2 increase since 1900

$M_{AT}(t)$: mass of carbon in the atmosphere, GtC

$M_{UP}(t)$: mass of carbon in the upper ocean, GtC

$M_{LO}(t)$: mass of carbon in the lower ocean, GtC

$\Omega(t)$: damage function; percentage of unusable output due to harms from climate change

$\Lambda(t)$: abatement cost function; percentage of output spent on reducing emissions

$\mu(t)$: emissions reduction rate as a percent of total emissions (set to zero unless emissions controls are imposed)

3 Parameters

t: time in decades from 2005–2014, 2015–2024, ..., 2595–2604. The last time period is denoted $Tmax$ and the series is 0-indexed

$L(t)$: global population, millions

$A_g(t)$: growth rate of total factor productivity

$\sigma(t)$: ratio of uncontrolled industrial emissions to output, tC per output in 2005 US dollars

$\sigma_g(t)$: rate of decline of carbon intensity per decade, expressed as a *positive* number

$E_{Land}(t)$: carbon emissions from land use (ie, deforestation), GtC per period

$F_{EX}(t)$: radiative forcing due to other GHGs, W/m^2 increase since 1900

$BC(t)$: cost of backstop technology, thousands of 2005 US dollars per ton of CO_2

3.1 Parameters that users choose

α: elasticity of marginal utility of consumption. $\alpha \in [1, 3]$. Default = 1.5.

ρ: social time preference rate. $\rho \in [0, .1]$. Default = 0.015.

$L(Tmax)$: asymptotic population in the last period. $L(Tmax) \in [8000, 12000]$. Default = 8700.

Δ_a: decline rate of technological change. $\Delta_a \in [0.05, 1.5]$. Default = 0.9.

δ_K: depreciation rate of technological change. $\delta_k \in [0.08, 0.2]$. Default = 0.1.

σ_{d1}: decline rate of decarbonization. $\sigma_{d1} \in [0, 0.06]$. Default = 0.006.

ϵ: damage exponent in climate damage function. $\epsilon \in [1, 4]$. Default = 2.

$T_{2\times CO_2}$: temperature increase (°C) from a doubling of preindustrial CO_2. $T_{2\times CO_2} \in [1, 5]$. Default = 3.2.

BC_g: cost decline in backstop technology in percent. $BC_g \in [0, 0.2]$. Default = 0.05.

θ_2: exponent of emission reduction rate in abatement cost function. $\theta_2 \in [2, 4]$. Default = 2.8.

$CCum$: fossil fuels remaining, measured in CO_2 emissions; maximum consumption of fossil fuels (GtC). $CCum \in [6000, 9000]$. Default = 6000.

s: savings rate. $s \in [0.15, 0.25]$. Default = 0.22.

User-defined climate treaty

$ecap(5)$: the mandated decrease in emissions by 2050 as a share of 2005 year emissions; emissions cap by 2050. $ecap(5) \in [0, 1]$. Default = 0.

$ecap(10)$: the mandated decrease in emissions by 2100 as a share of 2005 year emissions; emissions cap by 2100. $ecap(10) \in [0, 1]$. Default = 0.

$ecap(15)$: the mandated decrease in emissions by 2150 as a share of 2005 year emissions; emissions cap by 2150. $ecap(15) \in [0, 1]$. Default = 0.

$\varphi(5)$: fraction of emissions under control in 2050. $\varphi(5) \in [0, 1]$. Default = 1

$\varphi(10)$: fraction of emissions under control in 2100. $\varphi(10) \in [0, 1]$. Default = 1

$\varphi(15)$: fraction of emissions under control in 2150. $\varphi(15) \in [0, 1]$. Default = 1

User-defined carbon tax

$\tau(5)$: tax in 2005 US dollars per ton of carbon emissions in 2050. $\tau(5) \in [0, 1]$. Default = 0

$\tau(10)$: tax in 2005 US dollars per ton of carbon emissions in 2100. $\tau(10) \in [0, 1]$. Default = 0

$\tau(15)$: tax in 2005 US dollars per ton of carbon emissions in 2150. $\tau(15) \in [0, 1]$. Default = 0

3.2 Parameters from data

3.2.1 Economics model

$L(0)$: 2005 world populations in millions = 6411

L_g: growth rate of population per decade = 0.5

$A(0)$: inital level of total factor productivity = 0.0303220

$A_g(0)$: initial growth rate of TFP per decade = 0.16

Δ_b: decline rate of the decline of growth of productivity = 0.2

γ: capital elasticity of output in production function = 0.300

$Q(0)$: 2005 world gross output = 55.34

$K(0)$: 2005 capital value = 137

$\sigma(0)$: 2005 effective carbon intensity = 0.14452

$\sigma_g(0)$: initial rate of decline of carbon intensity per period = 0.158

$E_{Land}(0)$: carbon emissions from deforestation in 2005 = 1.1

$E(0)$: total emissions in year 2005 = 84.1910

$BC(0)$: cost of backstop technology in 2005 = 1.26

π_2: coefficient on the damage exponent term, $T_{AT}(t)^{\epsilon}$ in climate damage function = 0.0028.

3.2.2 Climate model

$M_{AT}(0)$: mass of carbon in the atmosphere in 2005 = 787

$M_{UP}(0)$: mass of carbon in the upper ocean in 2005 = 1600

$M_{LO}(0)$: mass of carbon in the lower ocean in 2005 = 10100

M_{PI}: preindustrial concentration of carbon in the atmosphere = 592

ϕ: transition matrix for carbon, where ϕ_{ij} is the transfer rate of carbon from reservoir i to reservoir j for i, $j = AT$, UP, and LO:

$$\begin{bmatrix} 0.88 & .12 & 0 \\ 0.04704 & 0.94796 & .005 \\ 0 & 0.00075 & 0.99925 \end{bmatrix}$$

$F_{2\times CO_2}$: forcing from a doubling of preindustrial $CO_2 = 3.8$

ξ_1: inverse of thermal capacity of the atmosphere and the upper ocean $= .220$

ξ_2: ratio of the thermal capacity of the deep oceans to the transfer rate from the shallow ocean to the deep ocean $= 0.310$

ξ_3: transfer rate of heat from the upper ocean to the deep ocean $= 0.050$

$T_{AT}(0)$: temperature change from 1900 until 2000 $= 0.83$

$T_{LO}(0)$: temperature change in the lower ocean from 1900 until 2000 $= 0.0068$

$F_{EX}(0)$: non-CO_2 forcings in 2005 $= 0.83$

$F_{EX}(10)$: estimate of non-CO_2 forcings in 2100 $= 0.30$

Linear Temperature Model

ψ: temperature increase (°C)per trillion tons of carbon emissions $= 2$

BEAM Carbon Model

Initial conditions:

$M_{AT}(0)$: mass of carbon in the atmosphere in 2005 $= 808.9$

$M_{UP}(0)$: mass of carbon in the upper ocean in 2005 $= 725$

$M_{LO}(0)$: mass of carbon in the lower ocean in 2005 $= 35{,}641$

$pH = 8.17$

Initial transition matrix for carbon, where ϕ_{ij} is the transfer rate of carbon from reservoir i to reservoir j for i, $j = AT$, UP and LO:

$$\begin{bmatrix} -0.2 & 0.2 & 0 \\ 0.2 & -0.2 & .05 \\ 0 & 0.001 & -.001 \end{bmatrix}$$

k_a: inverse exchange timescale between the atmosphere and the ocean (yrs^{-1}) $= 0.2$

k_d: inverse exchange timescale between upper and lower ocean (yrs^{-1}) $= 0.05$

δ: ratio of volume in lower and upper ocean = 50

k_H: ratio of the molar concentrations of CO_2 in atmosphere and ocean = 1.23×10^3

k_1: dissociation coefficient (mol/kg) = 8.0×10^{-7}

k_2: dissociation coefficient (mol/kg) = 4.53×10^{-10}

AM: number of moles in the atmosphere = 1.77×10^{20}

OM: number of moles in the ocean = 7.8×10^{22}

Alk: alkalinity (GtC) = 767